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# The boundary layer on a flat plate moving transversely in a rotating, stratified fluid

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The motion of a horizontal plate moving in its own plane in a rotating, stratified fluid is studied to establish the parameter conditions specifying the onset of boundary-layer blocking for the entire range of Rossby and Russell numbers. Régimes in Rossby–Russell number space defining the range of applicability of the inertia–viscous, buoyancy–viscous, and Coriolis–viscous boundary-layer balances are presented, and similarity solutions valid over a limited region of that space are derived. The plate drag and heat transfer are computed from the similarity solutions.

## 1. Introduction

Kelly & Redekopp (1970) and Redekopp (1970) studied the motion of a finite horizontal plate in a stratified, non-rotating fluid. The results obtained from those analyses showed that stratification significantly affects the plate drag, and indeed the entire flow structure, when the Russell number is greater than unity. Further, the boundary layer on the plate was shown to be blocked; that is, an upstream viscous wake and a boundary layer whose thickness decreases in the downstream direction appears, when the Russell number exceeds a critical power of the Reynolds number, the exponent depending in an essential way on the Prandtl number. In the present study the coupled influence of fluid rotation and stratification on the boundary-layer-blocking condition and the frictional drag and heat-transfer characteristics of a horizontal plate are investigated.

Long (cf. Martin 1966) showed that the blocking of a horizontal boundary layer in a stratified flow could be most clearly understood on the basis of a vorticity balance in the boundary layer. The additional vorticity generated via the Coriolis forces in a rotating, stratified flow will alter the vorticity balance which led to the results reported in Kelly & Redekopp (1970) and Redekopp (1970), providing the possibility of a modified or extended blocking condition.

## 2. Formulation

We consider a thin, horizontal plate of length $L$ moving in its own plane with constant velocity $U_0$ in a stratified fluid rotating about a vertical axis with constant angular speed $\Omega$. The fluid is bounded in the vertical by two infinite

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horizontal planes, also rotating at frequency $\Omega$, and the stratification in the uniformly rotating state is statically stable and prescribed by a linear variation of temperature with altitude

$$T_s = T_0(1+\beta_0 x_3) \quad (\beta_0 > 0). \tag{2.1}$$

$T_0$ defines the undisturbed fluid temperature at the altitude of the plate and $\beta_0^{-1}$ is the scale height of the stratification. The above temperature distribution corresponds only to an approximate equilibrium solution of the equations of

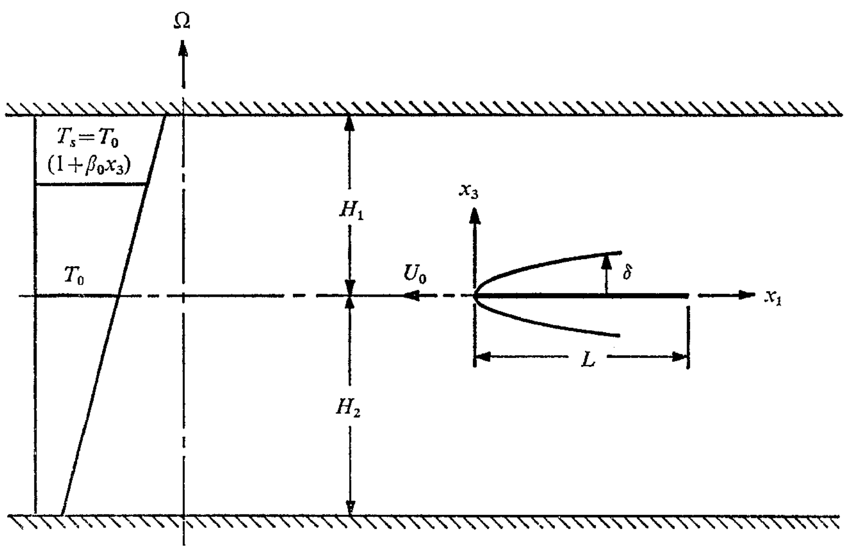


FIGURE 1. A schematic diagram of the flow model.

motion for a rotating fluid valid when the quantity $(\Omega^2 L/g)$ is small compared to unity (cf. Greenspan 1968, p. 13), a restriction we assume to be satisfied throughout. We describe the thermodynamic state of the fluid by the equation

$$\rho_0 = \rho[1-\alpha_0(T-T_0)], \tag{2.2}$$

where $\alpha_0$ denotes the (constant) coefficient of thermal expansion. The temperature $T_w$ of the moving plate is taken to be constant, but different from the temperature $T_0$ of the undisturbed fluid at the same altitude. A schematic representation of the flow model appears in figure 1.

The equations of motion with the velocity measured relative to a frame rotating with angular velocity $\Omega$ are (cf. Greenspan 1968, p. 16)

$$\nabla \cdot \mathbf{q} = 0, \tag{2.3}$$

$$\frac{\partial \mathbf{q}}{\partial t^*} + \mathbf{q}\cdot\nabla\mathbf{q} + \frac{2}{Ro}\hat{k} \wedge \mathbf{q} = -\nabla\pi^* + \frac{\alpha\theta}{F} T^*\hat{k} + \frac{1}{R}\nabla^2\mathbf{q} \tag{2.4}$$

and

$$\left[\frac{\partial}{\partial t^*} + \mathbf{q}\cdot\nabla - \frac{1}{PR}\nabla^2\right]\left(T^* + \frac{\beta}{\theta}z\right) = 0. \tag{2.5}$$

In writing these equations we have invoked the Boussinesq approximation, which is consistent with the equation of state (2.2), and have defined dimensionless variables as

$$\left.\begin{aligned} t^* &= \frac{t}{L/U_0}, \quad (x,y,z) = \frac{(x_1,x_2,x_3)}{L}, \\ \mathbf{q} &= (u,v,w) = \frac{(v_1,v_2,v_3)}{U_0}, \\ \pi^* &= \frac{p-p_0}{\rho_0 U_0^2} - \frac{1}{2}\left(\frac{\Omega L}{U_0}\right)^2 (x^2+y^2) + \frac{z}{F}\left(1-\frac{\alpha\beta z}{2}\right), \\ \text{and} \qquad T^* &= \frac{T-T_s}{T_w-T_0} = \frac{T-T_0}{T_w-T_0} + \frac{\beta}{\theta} z. \end{aligned}\right\} \tag{2.6}$$

The quantities $\alpha, \beta$ and $\theta$ are dimensionless constants given by

$$\alpha = \alpha_0 T_0, \quad \beta = \beta_0 L \quad \text{and} \quad \theta = \frac{T_w - T_0}{T_0}. \tag{2.7}$$

The remaining parameters are the Froude number ($F = U_0^2/(gL)$), the Prandtl number ($P = \nu_0/\kappa_0$), the Rossby number ($Ro = U_0/(\Omega L)$), and the Reynolds number ($R = U_0 L/\nu_0$). Without loss of generality we take $\alpha$ equal to unity in the following analysis. The final results can be applied to fluids with arbitrary $\alpha$ if both $\beta$ and $\theta$ are replaced everywhere by $\alpha\beta$ and $\alpha\theta$, but the Boussinesq equations are only a valid first-order approximation to the complete equations of motion if $\alpha\beta$ and $\alpha\theta$ are small compared to unity (Mihaljan 1962). In what follows the ratio $\beta/\theta$ is assumed to be of order unity. No restrictions are imposed on the magnitudes of the remaining parameters except that $(\Omega^2 L/g)$ be small as mentioned earlier.

Transforming to a co-ordinate system fixed to the leading edge of the plate, assuming the plate to be infinitely wide so that the velocity can be expressed in terms of a stream function

$$\mathbf{q} = \hat{\jmath} \times \nabla\psi_{(x,z)} + \hat{\jmath} v = \hat{\imath}\frac{\partial\psi}{\partial z} + \hat{\jmath} v - \hat{k}\frac{\partial\psi}{\partial x}, \tag{2.8}$$

and eliminating the reduced pressure $\pi$ by taking the vector cross product of the momentum equation (2.4), the set of governing equations can be written in the form

$$\left[L_{(x,z,\psi)} - \frac{1}{R}\nabla^2\right]\nabla^2\psi - \frac{2}{Ro}v_z + \frac{\theta}{\beta} R u^2 T_x = 0, \tag{2.9}$$

$$\left[L_{(x,z,\psi)} - \frac{1}{R}\nabla^2\right] v + \frac{2}{Ro}(\psi_z - 1) = 0, \tag{2.10}$$

and

$$\left[L_{(x,z,\psi)} - \frac{1}{PR}\nabla^2\right] T - \frac{\beta}{\theta}\psi_x = 0. \tag{2.11}$$

The first term in the bracket expressions represents the non-linear advective operator defined as

$$L_{(x,z,\psi)} = \psi_z \frac{\partial}{\partial x} - \psi_x \frac{\partial}{\partial z}, \tag{2.12}$$

the symbols in parentheses denoting the independent and dependent variables of the operator. The asterisk has been deleted from the perturbation temperature in (2.9) and (2.11), since it is clearly dimensionless. The Russell number defined by (cf. Miles 1969)

$$Ru^2 = \frac{\beta}{F} = \frac{g\beta_0 L^2}{U_0^2} \tag{2.13}$$

has been introduced in the vorticity equation (2.9). The quantity $(g\beta_0)^{\frac{1}{2}}$ is the intrinsic or Brunt–Väisälä frequency, which is a constant for the linear stratification model and the Boussinesq approximation.

Boundary conditions to be used in the solution of the foregoing equations are the no-slip conditions at the plate and boundaries. On the plate these conditions are given by

$$v_{(x,0)} = \psi_{(x,0)} = \psi_z(x,0) = 0 \quad \text{and} \quad T_{(x,0)} = 1 \quad (0 \leqslant x \leqslant 1). \tag{2.14}$$

On the boundaries located at $z = \pm H/L$, the conditions are

$$v = \psi = T = 0 \quad \text{and} \quad \psi_z = 1. \tag{2.15}$$

In the following analysis we take $H \geqslant O(L)$, so that, as far as the boundary-layer solutions are concerned, the latter conditions can be applied at $z$ approaching plus or minus infinity. Further, solutions are presented only for the boundary layer on the upper surface of the plate, since, in the Boussinesq limit, the lower-boundary-layer solution can be obtained by a simple reflexion.

## 3. The boundary-layer approximation

The approximate form of (2.9)–(2.11), describing the motion in thin boundary layers surrounding the plate, is systematically derived by introducing the transformations,

$$\psi_{(x,z)} = \epsilon\Psi_{(x,\zeta)}, \tag{3.1}$$

$$v_{(x,z)} = \sigma V_{(x,\zeta)}, \tag{3.2}$$

and

$$T_{(x,z)} = \Phi_{(x,\zeta)}, \tag{3.3}$$

where

$$\zeta = z/\epsilon, \quad \epsilon \sim \delta/L \ll 1. \tag{3.4}$$

The stream-function transformation satisfies the requirement that the streamwise velocity match uniformly between the boundary layer and outer flow and the parameter $\sigma$ is required to maintain a proper balance between the Coriolis and viscous terms in the lateral momentum equation (2.10). Motion in the lateral direction is driven solely by the Coriolis force, a requirement which is sufficient to determine the form of $\sigma$. The parameter $\epsilon$ is determined by requiring the coefficient of the highest-order viscous term in (2.9), governing the streamwise motion, to be unity, while all other terms in the equation are of order unity or smaller, and the temperature transformation is dictated by the boundary conditions.

In carrying out the foregoing transformation it is convenient to write the Prandtl, Rossby, and Russell numbers as a power of the Reynolds number,

$$P = R^d, \quad Ro = R^c \quad \text{and} \quad Ru^2 = R^b, \tag{3.5}$$

so that, in effect, only one parameter appears in the governing system of equations The exponents $d, c$ and $b$ depend on the relative magnitudes of the respective parameters as compared with the Reynolds number.

Substituting relations (3.1)–(3.4) into (2.9)–(2.11) and using (3.5) yields the boundary-layer equations

$$\left[L_{(x,\zeta,\Psi)} - \frac{\epsilon^{-2}}{R}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\Psi - \frac{2\sigma}{R^c}V_\zeta + \epsilon\frac{\theta}{\beta}R^b\Phi_x = 0, \quad (3.6)$$

$$\left[L_{(x,\zeta,\Psi)} - \frac{\epsilon^{-2}}{R}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]V + \frac{2}{\sigma R^c}(\Psi_\zeta - 1) = 0 \quad (3.7)$$

and

$$\left[L_{(x,\zeta,\Psi)} - \frac{\epsilon^{-2}}{R^{1+d}}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]\Phi - \epsilon\frac{\beta}{\theta}\Psi_x = 0. \quad (3.8)$$

Notice that stratification has at most a second-order influence on heat diffusion when $\beta/\theta$ is of order unity or smaller, and that the baroclinic generation of vorticity depends on the Prandtl number via the coupling of the vorticity and energy equations. If the Prandtl number is large ($\epsilon_T = (PR)^{-\frac{1}{2}} \ll \epsilon$), heat diffusion can be neglected so far as the velocity field is concerned, and the solution of (3.8) is

$$\Phi_{(x,\zeta)} = \epsilon(\beta/\theta)(\Psi_{(x,\zeta)} - \zeta). \quad (3.9)$$

### 3.1. *The inertia–viscous boundary-layer balance*

When the advective terms dominate over the Coriolis and buoyancy terms in (3.6), $\epsilon$ takes on the familiar value

$$\epsilon = R^{-\frac{1}{2}}, \quad (3.10)$$

and, from (3.7), $\sigma$ is given by

$$\sigma = R^{-c} = Ro^{-1}. \quad (3.11)$$

To derive the latter result we applied the criterion that the lateral motion is driven by the Coriolis force and the boundary conditions require that the highest-order viscous term in (3.7) be retained to first order. Examining (3.6), we find that the inertia–viscous balance with scale (3.10) defines the boundary layer on the plate, provided the conditions

$$\left.\begin{aligned} &\sigma/R^c < 1, \quad \text{i.e.} \quad Ro > 1 \quad (c > 0) \\ \text{and} \quad &\epsilon R^b < 1, \quad \text{i.e.} \quad Ru^2 < R^{\frac{1}{2}} \quad (b < \tfrac{1}{2}) \end{aligned}\right\} \quad (3.12a, b)$$

are satisfied. If the Prandtl number is sufficiently large that heat diffusion is at most of second order on the scale (3.10) (that is, $P > R^{\frac{1}{2}}$), the temperature field can be approximated by (3.9) and the latter condition becomes

$$\epsilon^2 R^b < 1, \quad \text{i.e.} \quad Ru^2 < R \quad (b < 1). \quad (3.13)$$

When the conditions prescribed by (3.12$a, b$) apply, the system (3.6)–(3.8) can be solved approximately by means of perturbation similarity expansions of the form

$$\left.\begin{aligned} \Psi_{(x,\zeta)} &= x^{\frac{1}{2}}f_{1(\eta)} + R^{b-\frac{1}{2}}xf_{b(\eta)} + R^{-2c}x^{\frac{5}{2}}f_{c(\eta)} + O(\epsilon), \\ V_{(x,\zeta)} &= xg_{1(\eta)} + R^{b-\frac{1}{2}}x^{\frac{3}{2}}g_{b(\eta)} + R^{-2c}x^3g_{c(\eta)} + O(\epsilon), \\ \text{and} \quad \Phi_{(x,\zeta)} &= h_{1(\eta)} + R^{b-\frac{1}{2}}x^{\frac{1}{2}}h_{b(\eta)} + R^{-2c}h_{c(\eta)} + O(\epsilon), \end{aligned}\right\} \quad (3.14a\text{–}c)$$

where the similarity variable $\eta$ is given by

$$\eta = \zeta/x^{\frac{1}{2}}. \tag{3.15}$$

The second and third terms in the expansion represent the contributions to the velocity and temperature fields arising from the baroclinic generation of vorticity and the Coriolis forces respectively. In writing the above expansions we have chosen to neglect second-order influences arising from boundary-layer-induced corrections to the outer flow, and have assumed that the baroclinic and Coriolis terms are larger than order $\epsilon$ but smaller than order unity. Hence, the equations and solutions subsequently derived are applicable in the parameter ranges

$$0 < b < \tfrac{1}{2} \quad \text{or} \quad 1 < Ru^2 < R^{\frac{1}{2}} \tag{3.16}$$

and

$$0 < c < \tfrac{1}{4} \quad \text{or} \quad 1 < Ro < R^{\frac{1}{2}}. \tag{3.17}$$

For $b \leqslant 0$ and $c \geqslant \frac{1}{4}$ the inertia–viscous balance still applies, but then terms of order $\epsilon$ must be considered and account must be taken of the displacement-induced velocity in the outer flow.

Substituting the above similarity expansions into the governing set of equations and collecting terms of equal order leads to the following system of differential equations and corresponding boundary conditions:

$$\left.\begin{aligned} &f_1''' + \tfrac{1}{2}f_1 f_1'' = 0, \quad f_{1(0)} = f'_{1(0)} = 0, \quad f'_{1(\infty)} = 1,\\ &g_1'' + \tfrac{1}{2}f_1 g_1' + f_1' g_1 = 2(f_1' - 1), \quad g_{1(0)} = g_{1(\infty)} = 0,\\ &h_1'' + \tfrac{1}{2}P f_1 h_1' = 0, \quad h_{1(0)} = 1, \quad h_{1(\infty)} = 0; \end{aligned}\right\} \tag{3.18a–c}$$

$$\left.\begin{aligned} &f_b''' + \tfrac{1}{2}f_1 f_b'' - \tfrac{1}{2}f_1' f_b' + f_1'' f_b = -\frac{1}{2}\frac{\theta}{\beta}\left[\eta h_1 + \int_\eta^\infty h_1\, d\eta\right], \quad f_{b(0)} = f'_{b(0)} = f'_{b(\infty)} = 0,\\ &g_b'' + \tfrac{1}{2}f_1 g_b' - \tfrac{3}{2}f_1' g_b = g_1 f_b' - g_1' f_b + 2f_b', \quad g_{b(0)} = g_{b(\infty)} = 0,\\ &h_b'' + P\{\tfrac{1}{2}f_1 h_b' - \tfrac{1}{2}f_1' h_b + h_1' f_b\} = \tfrac{1}{2}\delta_{(b)} P \frac{\beta}{\theta}(\eta f_1' - f_1), \quad h_b(0) = 0,\\ &h_{b(\infty)} = -1{\cdot}72\frac{\beta}{\theta}\delta_{(b)}; \end{aligned}\right\} \tag{3.19a–c}$$

and

$$\left.\begin{aligned} &f_c''' + \tfrac{1}{2}f_1 f_c'' - 2f_1' f_c' + \tfrac{5}{2}f_1'' f_c = -g_1, \quad f_{c(0)} = f'_{c(0)} = f'_{c(\infty)} = 0,\\ &g_c'' + \tfrac{1}{2}f_1 g_c' - 3f_1' g_c = g_1 f_c' - \tfrac{5}{2}g_1' f_c + 2f_c', \quad g_{c(0)} = g_{c(\infty)} = 0,\\ &h_c'' + P\{\tfrac{1}{2}f_1 h_c' - 2f_1' h_c + \tfrac{5}{2}h_1' f_c\} = 0, \quad h_{c(0)} = h_{c(\infty)} = 0. \end{aligned}\right\} \tag{3.20a–c}$$

The solutions of (3.18*a*) and (3.18*c*) are well known (cf. Schlichting 1968), where $f_1$ is the dimensionless Blasius stream function and $h_1$ is the Pohlhausen temperature function. Equations (3.19*a*) and (3.19*c*) were solved for $b = 0$ in Redekopp (1970), but a numerical error has subsequently been detected in those solutions. The corrected solutions, together with the solutions of the remaining equations in the above set, are presented in § 5 of this paper. The modification required in the higher-order energy equation (3.19*c*) when $b = 0$ is included by use of the $\delta$-function notation. Notice that the boundary condition at infinity must also be modified as indicated for that condition. That the revised boundary condition is necessary can be seen from the asymptotic solution of (3.19*c*), and, as shown in

Redekopp (1970) it matches identically to the outer flow when the displacement effect of the $f_1$ solution is taken into account. When $b = 0$ the boundary condition $f'_{b(\infty)}$ should also be corrected to include the displacement-induced velocity in the outer flow. However, its value is very difficult to compute, since the displacement distribution on the plate *and in the downstream wake* must be known in order to make the calculation. For the sake of approximation, if one makes the calculation assuming the plate is semi-infinite, $f'_{b(\infty)}$ vanishes identically (see Kelly & Redekopp 1970). The calculations presented later for $b = 0$ are made on the basis of the latter approximation. Similarity is not possible for the rotational correction to the temperature field (see (3.20*c*)) when $c = \frac{1}{4}$, so the solution for $h_c$ applies for $0 < c < \frac{1}{4}$, although $f_c$ and $g_c$ apply equally for $c = \frac{1}{4}$.

### 3.2. *The buoyancy–viscous boundary-layer balance*

We now examine the case when the baroclinic generation term dominates over the advective and Coriolis effects in (3.6). Applying the criterion for determining $\epsilon$ and $\sigma$ yields

$$\epsilon = R^{-\frac{1}{3}(1+b)} = (R\,Ru^2)^{\frac{1}{3}} \tag{3.21}$$

and

$$\sigma = R^{\frac{1}{3}(1-2b-3c)} = \left(\frac{R}{Ru^4Ro^3}\right)^{\frac{1}{3}}. \tag{3.22}$$

One can show directly that this scaling applies when

$$b > \tfrac{1}{2},\quad (Ru^2 > R^{\frac{1}{2}})\quad \text{and}\quad c > \tfrac{1}{3}(1-2b),\quad (Ro > R^{\frac{1}{3}}Ru^{-\frac{2}{3}}). \tag{3.23}$$

A restriction on $d$ can be derived by considering the boundary-layer scaling for large Prandtl numbers.

When the temperature field is approximated by (3.9) (valid when $P \gg 1$), the boundary-layer transformation requires

$$\left.\begin{aligned} \epsilon &= R^{-\frac{1}{4}(1+b)} = (R\,Ru^2)^{-\frac{1}{4}} \\ \text{and}\quad \sigma &= R^{\frac{1}{2}(1-b-2c)} = \frac{R^{\frac{1}{2}}}{Ru\,Ro}, \end{aligned}\right\} \tag{3.24$a,b$}$$

showing that the Prandtl number affects the scale of the velocity boundary layer through the baroclinic term when buoyancy has a first-order effect. Using the above scaling, the boundary-layer equations have the form

$$\left.\begin{aligned} \left[R^{\frac{1}{2}(1-b)}L_{(x,\zeta,\Psi)} - \left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\Psi - 2R^{1-b-2c}V_\zeta + \Psi_x &= 0, \\ \left[R^{\frac{1}{2}(1-b)}L_{(x,\zeta,\Psi)} - \left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]V + 2(\Psi_\zeta - 1) &= 0, \\ \text{and}\quad \left[L_{(x,\zeta,\Psi)} - R^{\frac{1}{2}(b-1-2d)}\left(\epsilon^2\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial\zeta^2}\right)\right]\left(\Phi + \epsilon\frac{\beta}{\theta}\zeta\right) &= 0. \end{aligned}\right\} \tag{3.25$a$–$c$}$$

This scaling applies for the parameter conditions

$$\left.\begin{aligned} &b > 1, \quad (Ru^2 > R), \\ &c > \tfrac{1}{2}(1-b), \quad (Ro > R^{\frac{1}{2}}Ru^{-1}), \\ \text{and}\quad &d > \tfrac{1}{4}(3b-1), \quad (P > Ru^{\frac{3}{2}}R^{-\frac{1}{4}}). \end{aligned}\right\} \tag{3.26}$$

The condition on $d$ arises from the requirement that heat diffusion be of order $\epsilon$ or smaller on the scale (3.24$a$) for the temperature to be given by (3.9). This defines the Prandtl number range for which the scaling (3.21), (3.22) is applicable.

In the remainder of §3 we consider only the non-diffusive velocity field, since no similarity solutions for the diffusive case are possible with $\beta/\theta = O(1)$. The non-diffusive velocity field is obtained by solving (3.25$a$) and (3.25$b$) with the temperature field given subsequently by (3.9), aside from a thin thermal diffusion layer near the boundary, which we neglect, since it has no effect on the velocity field to second order in $\epsilon$. Martin & Long (1968) discussed this thermal layer for the non-rotating case.

Equations (3.25$a, b$) can be solved approximately by expanding the dependent variables as

$$\begin{Bmatrix}\Psi \\ V\end{Bmatrix} = \begin{Bmatrix}\Psi^{(1)} \\ V^{(1)}\end{Bmatrix} + R^{1-b-2c}\begin{Bmatrix}\Psi^{(2)} \\ V^{(2)}\end{Bmatrix} + R^{\frac{1}{2}(1-b)}\begin{Bmatrix}\Psi^{(3)} \\ V^{(3)}\end{Bmatrix} + O(\epsilon^2), \tag{3.27}$$

where the second and third terms on the right-hand side represent the contributions of rotation and advection (respectively) to the boundary-layer velocity field. The most important feature of this boundary-layer balance pertains to the character of the first-order stream function $\Psi^{(1)}$. The parabolic equation for $\Psi^{(1)}$ was first derived by Long (1959), who showed (see Martin 1966) that no solutions are possible unless the sense of the time-like variable ($x$) is reversed. This indicates that the flow is now blocked, and an upstream viscous wake appears, which was studied by Long (1959, 1962), Martin & Long (1968) and Pao (1968). Furthermore, Martin & Long (1968) showed that the solution for $\Psi^{(1)}$ is uniformly valid, so that there is no stream displacement and the outer flow remains a uniform parallel flow. Thus, the next-order correction in (3.27) is of order $\epsilon^2$, accounting for axial viscous diffusion.

Introducing the change of variable

$$\xi = (1-x), \tag{3.28}$$

so that the horizontal co-ordinate is measured upstream from the trailing edge of the plate, and using the similarity transformations defined by

$$\left.\begin{aligned} \Psi_{(x,\zeta)} &= \xi^{\frac{1}{4}}F_{1(\eta)} + R^{1-b-2c}\xi^{\frac{5}{4}}F_{r(\eta)} + R^{\frac{1}{2}(1-b)}\xi^{-\frac{1}{4}}F_{a(\eta)} \\ \text{and}\qquad V_{(x,\zeta)} &= \xi^{\frac{1}{2}}G_{1(\eta)} + R^{1-b-2c}\xi^{\frac{3}{2}}G_{r(\eta)} + R^{\frac{1}{2}(1-b)}G_{a(\eta)}, \end{aligned}\right\} \tag{3.29$a, b$}$$

where $\eta$ is given by

$$\eta = \zeta/\xi^{\frac{1}{4}}, \tag{3.30}$$

the following sets of equations are obtained from (3.25$a, b$):

$$\left.\begin{aligned} &4F_1^{\text{iv}} - \eta F_1' + F_1 = 0, \quad F_{1(0)} = F_{1(0)}' = F_{1(\infty)}''' = 0,\ F_{1(\infty)}' = 1, \\ &G_1'' = 2(F_1' - 1), \quad G_{1(0)} = G_{1(\infty)} = 0; \end{aligned}\right\} \tag{3.31$a, b$}$$

$$\left.\begin{aligned} &4F_r^{\text{iv}} - \eta F_r' + 5F_r = -8G_1'', \quad F_{r(0)} = F_{r(0)}' = F_{r(\infty)}' = F_{r(\infty)}''' = 0, \\ &G_r'' = 2F_r', \quad G_{r(0)} = G_{r(\infty)} = 0; \end{aligned}\right\} \tag{3.32$a, b$}$$

and

$$\left.\begin{aligned} &4F_a'''' - \eta F_a' = F_1 F_1''', \quad F_{a(0)} = F_{a(0)}' = F_{a(\infty)}' = 0, \\ &G_a'' = 2F_a' + \tfrac{1}{4}G_1' F_1 - \tfrac{1}{2}G_1 F_1' \quad G_{a(0)} = G_{a(\infty)} = 0. \end{aligned}\right\} \tag{3.33$a, b$}$$

These equations describe the flow in a boundary layer whose characteristic thickness $\delta$ decreases in the streamwise direction as $\delta \sim (1-x)^{\frac{1}{4}}$ with an upstream viscous wake. The solution of (3.31*a*), together with experimental verification of the derived velocity profile, is presented in Martin & Long (1968) and Pao (1968). The solution of the remaining equations is presented later in §5.

### 3.3. *The Coriolis–viscous boundary-layer balance*

A third boundary-layer structure is possible, which arises when the Coriolis and viscous terms in (3.6) form the dominant balance. Moore & Saffman (1969) elucidated the dynamics of the motion of a horizontal plate in this régime when the fluid is homogeneous. They discussed both the case in which the fluid is bounded (vertically) and that in which it is unbounded, and showed that the basic geostrophic flow over the plate depends crucially on the existence of horizontal boundaries in the vicinity of the moving plate.

Of foremost interest here is the parameter régime in which the Coriolis–viscous (Ekman) balance applies. The boundary-layer scale is seen to be given by

$$\epsilon = R^{\frac{1}{2}(c-1)} = (Ro/R)^{\frac{1}{2}} = E^{\frac{1}{2}} \quad \text{and} \quad \sigma = 1, \tag{3.34}$$

where $E$ is the Ekman number $(\nu_0/\Omega L^2)$. The requisite parameter conditions can be derived directly, showing that the first-order boundary layer is the Ekman layer whenever

$$c < 0, \quad (Ro < 1) \quad \text{and} \quad b < \tfrac{1}{2}(1-3c) \quad (Ru^2 < R^{\frac{1}{2}}/Ro^{\frac{3}{2}}). \tag{3.35}$$

When the Prandtl number is sufficiently large $(d > \frac{1}{2}(1-3c))$ that thermal diffusion contributes negligibly to the baroclinic generation of vorticity, the latter condition is changed to

$$b < (1-2c) \quad \text{or} \quad Ru^2 < R/Ro^2. \tag{3.36}$$

If the square of the Russell number exceeds the indicated values (e.g. by increasing the static stability), the Ekman layer is blocked, and the buoyancy–viscous balance applies with upstream boundary-layer growth and an upstream viscous wake.

We discuss briefly the case in which the horizontal boundaries are far removed from the plate ($H > L/E$, as shown by Moore & Saffman), and the basic inviscid flow is known *a priori*. The first-order velocity field is then given by the non-divergent Ekman-layer solution

$$\left.\begin{aligned} \Psi^{(1)}_{(x,\zeta)} &= \zeta - \tfrac{1}{2}e^{-\zeta}(\sin\zeta - \cos\zeta) - \tfrac{1}{2} \\ \text{and} \qquad V^{(1)}_{(x,\zeta)} &= e^{-\zeta}\sin\zeta. \end{aligned}\right\} \tag{3.37$a,b$}$$

The solution must be modified in regions of scale $E^{\frac{1}{2}}$ at the leading and trailing edges of the plate to account for the streamwise change in boundary condition at those points.

The thermal field can be calculated subsequently from the energy equation, which to first-order is given by

$$P\,Ro(1-e^{-\zeta}\cos\zeta)\,\Phi^{(1)}_x - \Phi^{(1)}_{\zeta\zeta} = 0. \tag{3.38}$$

An exact solution is difficult to obtain, but two limiting cases are easily solved. These are

$$\Phi^{(1)}_{(x,\zeta)} = 1-\operatorname{erf}\left\{\frac{\zeta(PRo)^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\right\}, \qquad (3.39a)$$

when ($PRo \ll 1$), and, when ($PRo \gg 1$),

$$\Phi^{(1)}_{(x,\zeta)} = 1-\frac{\gamma(\frac{1}{3},\lambda)}{\Gamma(\frac{4}{3})}, \qquad (3.39b)$$

where $\gamma(a,x)$ is the incomplete gamma function and $\lambda = (PRo/(9x))^{\frac{1}{3}}\zeta$. The diffusive temperature field is important, since it provides the mechanism for establishing a streamwise variation in the velocity. The streamwise growth of the thermal boundary layer causes a lateral component of vorticity to be generated through the baroclinic effect, which, in turn, affects the velocity field. When this vorticity becomes a first-order quantity, the Ekman layer is 'blocked', and the boundary layer transists to the structure described in §3.2, i.e. the buoyancy–viscous case.

### 3.4. *A summary of the boundary-layer régimes*

A convenient representation of the results of the scaling analysis presented in §§3.1–3.3 is obtained by portraying graphically the régimes of applicability of the three characteristic boundary-layer balances in Rossby–Russell number parameter space. The Rossby–Russell number plane (see figure 2) is divided into three regions with the heavy lines defining the transition boundaries between the various régimes. At the point of intersection of the three heavy lines, the advective, Coriolis, buoyancy, and viscous terms are all important in the first-order boundary-layer balance. The dashed lines define the left-hand boundary of the buoyancy-viscous régime when the Prandtl number is large enough to render the non-diffusive approximation valid to first-order. Heat diffusion clearly enhances the role of buoyancy and enlarges the régime of applicability of the buoyancy–viscous balance. Recall that in the buoyancy–viscous régime the boundary layer grows in the upstream direction and an upstream wake is present.

One of the important results of this analysis is the condition for blocking of the non-divergent Ekman layer. The Ekman layer domain is limited by the Russell number and in a manner which depends on the Prandtl number. The blocking derives from the baroclinic generation of vorticity and is quite different from the Ekman-layer blocking discussed by Barcilon & Pedlosky (1967), in which case the static stability becomes sufficiently large to inhibit the steady vertical velocity induced by the divergent Ekman layer. The condition derived by Barcilon & Pedlosky is shown by the dotted line in figure 2, which, in the present notation, is given by

$$Ru^2 < (RRo^3)^{-\frac{1}{2}}.$$

The first-order solutions in (3.14), (3.29) and (3.37) are valid in their entire respective domains. The cross-hatched regions on figure 2 indicate the domains where the second- and third-order terms in the similarity solutions derived previously are valid. Region $A$ defines the parameter space where expansion (3.14) is valid and region $B$ defines the space where (3.29) applies.

## 4. The outer flow

The first-order correction to the outer flow is now discussed for each of the three regions in the Rossby–Russell number parameter space shown in figure 2.

For the inertia–viscous régime ($c > 0$) the solution for the stream function and the temperature was obtained for the entire range of Russell number in Kelly & Redekopp (1970) and Redekopp (1970) for non-diffusive (diffusive) flow. The important result of those analyses is the existence of an intermediate layer of

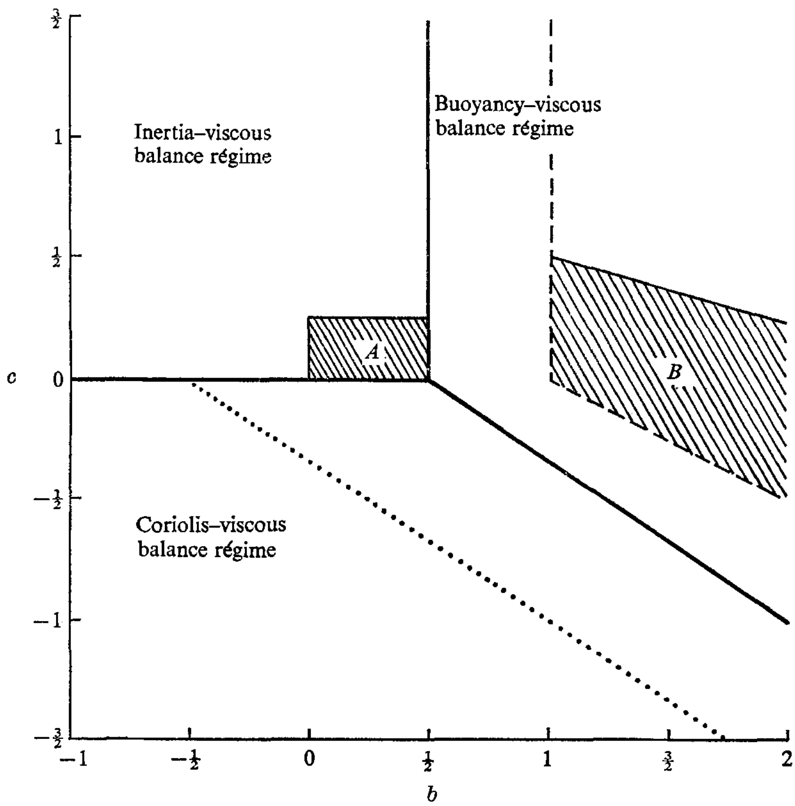


FIGURE 2. The characteristic boundary-layer balance régimes in Rossby–Russell number space.

scale $R^{-\frac{1}{2}b}$ for the range $0 < b < 1$ ($0 < b < \frac{1}{2}$) in non-diffusive (diffusive) flow, through which the Blasius boundary layer matches uniformly to the parallel external stream. Both independent variables in this intermediate layer must be scaled by the wavelength of internal waves moving with phase velocity $U_0$ and oscillating at the intrinsic frequency $(g\beta_0)^{\frac{1}{2}}$ rather than the plate length $L$. This result shows clearly that the condition for incipient blocking is

$$Ru^2 \sim R(Ru^2 \sim R^{\frac{1}{2}})$$

for non-diffusive (diffusive) flow, as derived from the boundary-layer analysis, rather than the much weaker condition, $Ru^2 \gg 1$, derived from a perturbation analysis of the unscaled equations (2.9)–(2.11); and it shows how the static stability decreases the vertical extent of the plate disturbance until it becomes of the order of the boundary-layer thickness and blocking occurs. Furthermore, it shows that streamwise variations become increasingly important, and that the complete elliptic equations must be considered as the blocking condition is approached. The required expansions and matching conditions for the stream function and temperature are given in Kelly & Redekopp (1970) and Redekopp (1970), while the solution for the lateral velocity can be obtained from the relation

$$v_{(x,z)} = (1/Ro)\,[\psi^{(1)}_{(x,z)} + \ldots], \tag{4.1}$$

where

$$\psi_{(x,z)} = z + \epsilon\psi^{(1)}_{(x,z)} + \ldots. \tag{4.2}$$

In the bouyancy-viscous balance régime the solution for the outer flow is trivial since the boundary-layer solution is uniformly valid. Thus, the outer flow remains that of a uniform, parallel stream. A similar conclusion arises for the Ekman layer régime when the plate is far removed ($H > L/E$) from other horizontal boundaries, since then the boundary layer is described by the non-divergent Ekman solution (3.37).

## 5. Numerical results

### 5.1. *Results for the inertia–viscous régime*

The effects of stratification and rotation on the boundary-layer flow in the inertia-viscous régime are characterized by the solution of equations (3.18)–(3.20). An important measure of these effects is their influence on the shear and heat transfer at the plate surface. Define dimensionless friction and heat transfer coefficients as

$$C_f^{(x,z)} = \frac{\tau_{xz}}{\rho_0 U_0^2}, \quad C_f^{(y,z)} = \frac{\tau_{yz}}{\rho_0 U_0^2} \quad \text{and} \quad C_h = \frac{q_w}{\rho_0 C_{p_0} U_0 (T_0 - T_w)}. \tag{5.1}$$

Then the following expressions for these quantities evaluated at the plate surface for the region of the inertia–viscous régime (designated by the cross-hatched area $A$ in figure 2) are obtained:

$$\left.\begin{aligned} C_f^{(x,z)} &= \frac{0{\cdot}332}{R_{x_1}^{\frac{1}{2}}}\left[1 + 3{\cdot}01\,\frac{Ru^2}{R^{\frac{1}{2}}}\,x^{\frac{1}{2}} f''_{b(0)} - 4{\cdot}22\,\frac{x^2}{Ro^2}\right], \\ C_f^{(z,y)} &= \frac{0{\cdot}190x^{\frac{1}{2}}}{Ro\,R^{\frac{1}{2}}}\left[1 + 5{\cdot}27\,\frac{Ru^2}{R^{\frac{1}{2}}}\,x^{\frac{1}{2}} g'_{b(0)} + 12{\cdot}47\,\frac{x^2}{Ro^2}\right], \\ C_h &= \frac{h'_{1(0)}}{PR_{x_1}^{\frac{1}{2}}}\left[1 + \frac{Ru^2}{R^{\frac{1}{2}}}\,x^{\frac{1}{2}}\,\frac{h'_{b(0)}}{h'_{1(0)}} + \frac{x^2}{Ro^2}\,\frac{h'_{c(0)}}{h'_{1(0)}}\right] + \frac{\beta/\theta}{PR}. \end{aligned}\right\} \tag{5.2a–c}$$

These expressions apply for either the upper or lower surface of the plate. The functions $f$, $g$ and $h$ are defined in (3.14), and $R_{x_1}$ denotes the Reynolds number based on the dimensional length $x_1$ measured from the leading edge of the plate in the streamwise direction. Numerical values for the shear and heat-flux functions appearing in the above expressions are shown graphically in figures 3 and 4 for a range of the stability parameter $\beta/\theta$ and the Prandtl number $P$ respectively. The results for $f''_{b(0)}$ and $h'_{b(0)}$ ($b = 0$) are corrected versions of the corresponding quantities presented in Redekopp (1970). The results shown are for a heated plate ($\beta/\theta > 0$), but can be extended directly for a cooled plate ($\beta/\theta < 0$) by simply reversing the sign of $f_b$, $g_b$, and $h_b$, as is clearly evident from (3.19).

Both the stability parameter $\beta/\theta$ and the Prandtl number have a strong influence on the shear and heat flux; the former through its direct appearance in (3.19$a$), (3.19$c$), and the latter through its influence on the first-order temperature field $h_1$, which affects $f_b$ (and consequently $g_b$ and $h_b$) via the right-hand side of (3.19$a$). (See Redekopp (1970) for a further discussion of the importance of these

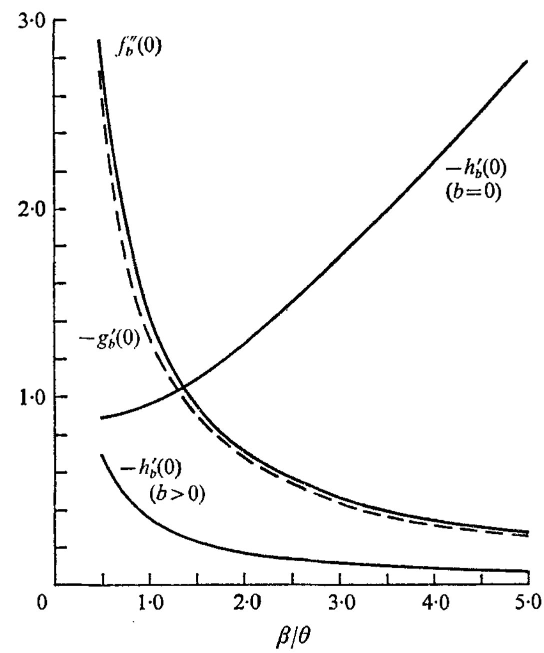


FIGURE 3. The shear and heat flux as a function of the stability parameter $\beta/\theta$ ($P = 1{\cdot}0$).

parameters on $f''_{b(0)}$ and $h'_{b(0)}$.) Stratification either aids or opposes the primary Blasius shear, depending on whether the plate is heated or cooled (respectively). Rotation is seen to always diminish the Blasius shear. Computing the total streamwise frictional drag on the plate per unit width yields

$$\tfrac{1}{2}C_D = \frac{0{\cdot}664}{R_{x_1}^{\frac{1}{2}}}\left[1 \pm 1{\cdot}51|f''_{b(0)}|\,\frac{Ru^2}{R^{\frac{1}{2}}} - \frac{0{\cdot}84}{Ro^2}\right], \quad \theta \gtrless 0. \tag{5.3}$$

Thus, for example, when $|\beta/\theta|$ and $P$ are unity and $Ru^2/R^{\frac{1}{2}}$ and $Ro^2$ are both a tenth, stratification increases (or decreases, depending on the sign of $\theta$) the drag by about 20 %, and rotation decreases the drag by approximately 8 %. Decreasing either the Prandtl number of $\beta/\theta$, or both, makes the effect of stratification more pronounced. Correspondingly, stratification changes the total heat transfer by 5 %, and rotation causes approximately a 9 % change.

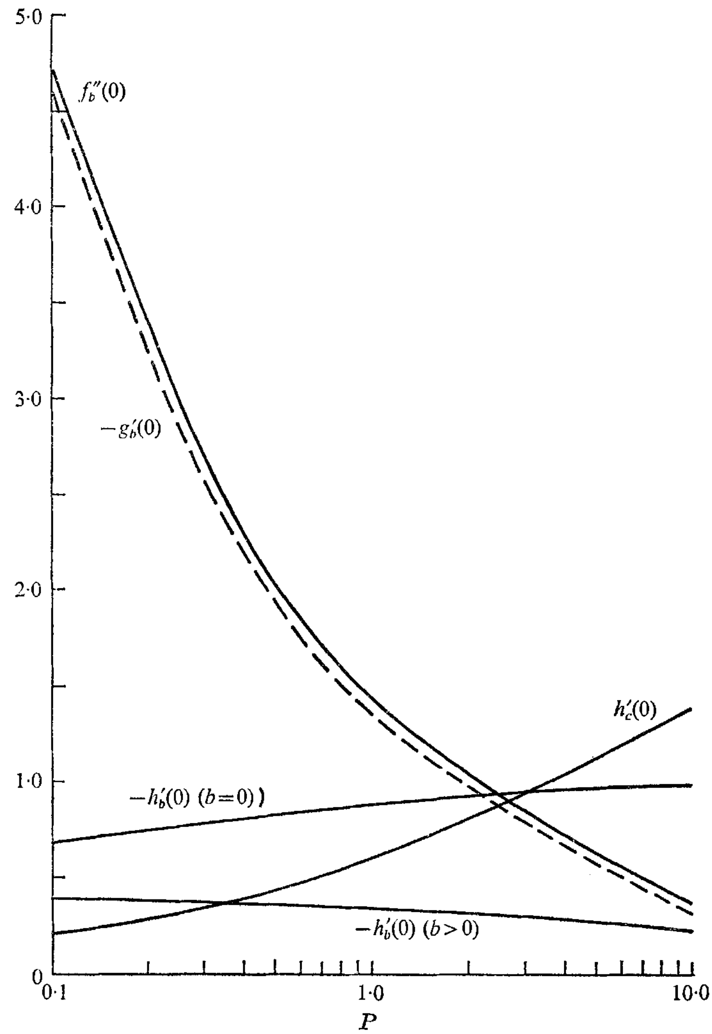


FIGURE 4. The shear and heat flux as a function of the Prandtl number ($\beta/\theta = 1{\cdot}0$).

The streamwise and lateral velocities are shown in figure 5. The total components can be computed from the relations

$$\left.\begin{aligned} u &= f'_{1(\eta)} + \frac{Ru^2}{R^{\frac{1}{2}}} x^{\frac{1}{2}} f'_{b(\eta)} + \frac{x^2}{Ro^2} f'_{c(\eta)} \\ \text{and} \qquad v &= \frac{1}{Ro}\left[g_{1(\eta)} + \frac{Ru^2}{R^{\frac{1}{2}}} x^{\frac{1}{2}} g_{b(\eta)} + \frac{x^2}{Ro^2} g_{c(\eta)}\right]. \end{aligned}\right\} \quad (5.4a, b)$$

Just as stratification and rotation significantly contribute to the frictional drag of the Blasius boundary layer, they will also influence the stability of the Blasius boundary layer through their modification of the mean velocity profile.

### 5.2. *Results for the non-diffusive buoyancy–viscous régime*

The frictional drag of the plate for the parameter region defined by the cross-hatched area $B$ on figure 2 is given by the expressions

$$\left.\begin{aligned} C_f^{(x,z)} &= \frac{1\cdot154}{(1-x)^{\frac{1}{4}}}\left(\frac{Ru^2}{R^3}\right)^{\frac{1}{4}}\left[1+0\cdot235(1-x)\frac{R}{(Ru\,Ro)^2}-\frac{0\cdot032}{(1-x)^{\frac{1}{2}}}\frac{R^{\frac{1}{2}}}{Ru}\right], \\ \text{and}\quad C_f^{(y,z)} &= \frac{0\cdot12}{Ro}\left(\frac{1-x}{Ru^2R}\right)^{\frac{1}{4}}\left[1+3.7(1-x)\frac{R}{(Ru\,Ro)^2}-\frac{66\cdot7}{(1-x)^{\frac{1}{2}}}\frac{R^{\frac{1}{2}}}{Ru}\right]. \end{aligned}\right\} \quad (5.5a, b)$$

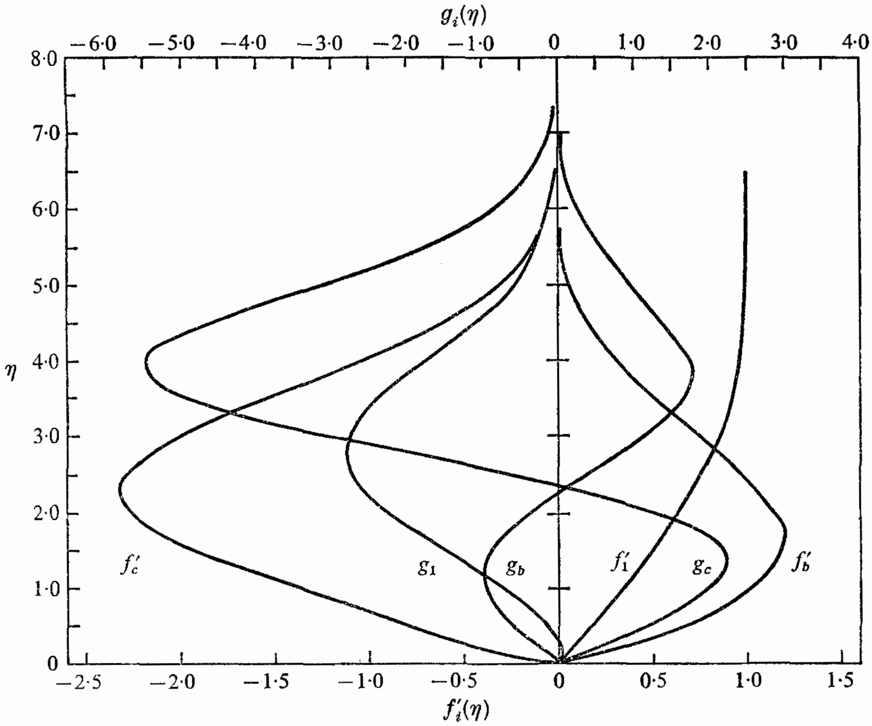


FIGURE 5. Velocity profiles for the inertia-viscous boundary-layer régime ($P$ = 1·0, $\beta/\theta$ = 1·0).

The second- and third-order terms in the brackets provide the higher-order corrections for the influence of rotation and advection, respectively. Rotation increases both the streamwise and lateral frictional coefficients over their first-order values while advection causes a decrease in each of the coefficients. Computing the total streamwise frictional drag on the plate per unit width yields the expression

$$\tfrac{1}{2}C_D = 1\cdot54\left(\frac{Ru^2}{R^3}\right)^{\frac{1}{4}}\left[1+0\cdot10\frac{R}{(Ru\,Ro)^2}-0\cdot006\frac{R^{\frac{1}{2}}}{Ru}\right]. \quad (5.6)$$

Including the effect of rotation increases the first-order total drag by, at most, 10 %, and the contribution from advection is always less than 1 %.

The streamwise velocity profiles are shown in figure 6. Interesting features of these profiles include the damped-wave-like structure and the smallness of the correction due to inertial effects. The first-order profile is discussed by Martin & Long (1968), who show that the asymptotic form of the solution for $F_1$ has the form of an exponentially damped wave. The corrections due to rotation and

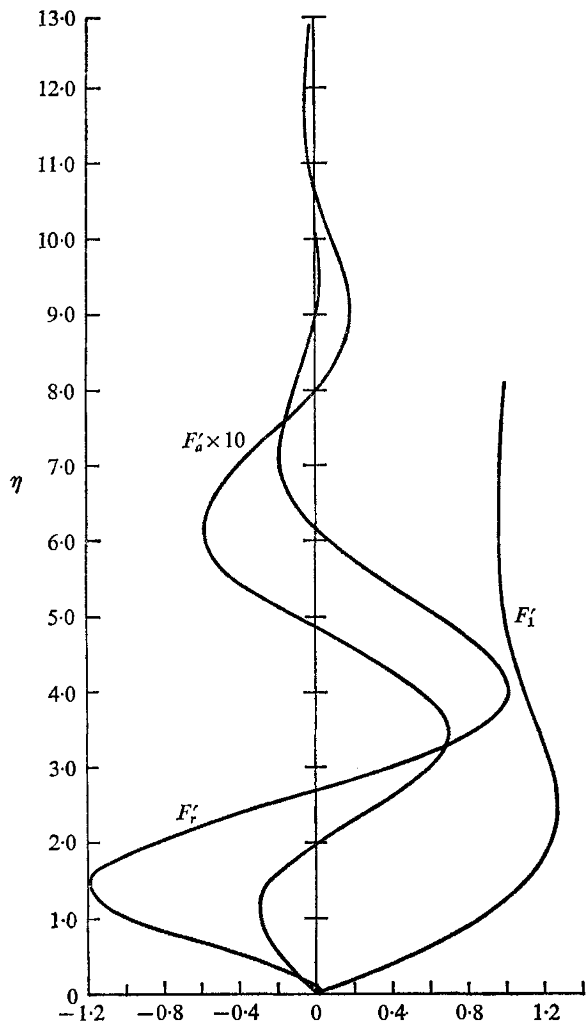


FIGURE 6. Streamwise velocity profiles for the buoyancy-viscous boundary-layer régime.

advection increase the vertical extent and waviness of the velocity field over that of the first-order distribution alone. However, the discrepancy between the theoretical and experimental velocity profiles in the outer region of the boundary layer reported by Martin & Long and by Pao (1968) must be due to the finiteness of the plate and the upstream wake disturbance, not the neglect of advection, since the advective correction is too small and in the wrong direction to yield the observed broad region of velocity defect at the outer edge of the boundary layer.

### 5.3. *Results for the Ekman régime*

For completeness, we present the shear and heat-flux coefficients for the Ekman régime as well. Using the prior definitions and (3.37) and (3.39) yields

$$C_f^{(x,z)} = C_f^{(y,z)} = \frac{1}{(RoR)^{\frac{1}{2}}} \tag{5.7}$$

for the frictional drag and

$$\left.\begin{aligned} C_h &= -\frac{1}{(\pi \times PR)^{\frac{1}{2}}} + \frac{\beta/\theta}{PR} \quad (PRo \ll 1),\\ C_h &= -\frac{(81\,x^2P^4R^3Ro)^{-\frac{1}{6}}}{\Gamma(\frac{4}{3})} + \frac{\beta/\theta}{PR} \quad (PRo \gg 1),\end{aligned}\right\} \tag{5.8a,b}$$

for the heat transfer. The first heat-transfer coefficient derives from the Oseen approximation to the energy and, consequently, is independent of the Rossby number. Note that it is a valid approximation even for Prandtl numbers of order unity when the Rossby number is small.

## 6. Summary

The description of the boundary layer on a horizontal plate in a stratified flow (Kelly & Redekopp (1970) and Redekopp (1970)) has been extended to include the influence of fluid rotation in the plane of the plate. The analysis extends the blocking criterion, derived in Kelly & Redekopp (1970) and Redekopp (1970) for all Russell numbers, to cover the entire range of Rossby numbers as well. Rotation modifies the former blocking criterion when the Rossby number is less than unity, extending the transition to higher Russell numbers. The parameter conditions specifying the onset of blocking are given by

$$Ru^2 \sim O\begin{Bmatrix} R^{\frac{1}{2}} \\ R^{\frac{1}{2}}/Ro^3 \end{Bmatrix} \quad \begin{matrix} Ro > 1, \\ Ro < 1, \end{matrix}$$

when $P < O(Ru^6/R)^{\frac{1}{4}}$, and

$$Ru^2 \sim O\begin{Bmatrix} R \\ R/Ro^2 \end{Bmatrix} \quad \begin{matrix} Ro > 1, \\ Ro < 1, \end{matrix}$$

when $P > O\,(Ru^6/R)^{\frac{1}{4}}$.

Solutions for the velocity and temperature fields in the inertia–viscous and buoyancy–viscous régimes show that the higher-order effects of stratification and rotation significantly influence the plate drag and heat transfer in the former régime, but that rotation and especially advection influence the flow characteristics negligibly in the latter régime.

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